$\mathbb{E}_\infty$-algebras.

We could have started the theory of derived rings over a field k by considering a different stable ∞-category as the home of our commutative algebra objects.

Recall Spctr has a $\otimes$-str, let $* \xrightarrow{\mathbb{1}_{Spctr}} Spctr.$ be its unit, by the universal property of Spc one has a colimit preserving functor:

$$\Sigma^\infty : Spc \rightleftarrows Spctr. : \Omega^\infty,$$ it has a right adjoint.

FACT: - Spctr has a t-structure described by:

$$Spctr^{\geq 1} := \{ X \in Spctr \mid \Omega^\infty(X) = * \},$$ Ω^∞ can also be described as the composite.

Equivalently, $Spctr^{\leq 0}$ is the essential image of Σ^∞ generated by under colimits & extensions.

$$\begin{tikzcd} Spctr = \lim(\cdots Spc_* \arrow[r, "\Omega"] & Spc_*) \arrow[d, "ev_0"] \arrow[dd, bend right, "\Omega^\infty"'] \\ & Spc_* \arrow[r, "oblv_*"] & Spc. \end{tikzcd}$$

- One can check that ~~for~~ $Spctr^\heartsuit \simeq Ab$ the ordinary category of abelian groups.

Exercise: [HA. Prop. 1.4.3.6.] Idea $X \in Spctr$ represented by $\{X(n)\}_{n\geq 0}$ w/ $X(n) = \Omega X(n+1)$.

- $\{X(n)\}_{n\geq 0} \in Spctr^{\geq 0} \iff X(n)$ is n-truncated. $\forall n \geq 0$.

- $\{X(n)\}_{n\geq 0} \in Spctr^{\leq 0} \iff X(n)$ is n-connected $\forall n \geq 0$.

$$\Rightarrow Spctr^\heartsuit = \lim \cdots \to EM_1(Spc) \to \underset{\substack{\shortparallel \\ \text{0-trun. \& 0-conn. objects of } Spc}}{EM_0(Spc)}.$$

(—).

HTT Prop. 7.2.2.12 identifies $EM_n(Spc). \xrightarrow[\sim]{n \geq 2.} \text{ComGrp.}(\tau_{\leq 0}(Spc))$.
b/c Spc is an ∞-topos.
So the limit stabilizes.
Thus, $\pi_n: Spctr \to Ab.$ and
$$Spctr^{\leq 0} = \{X \in Spctr \mid \pi_i(X) = 0 \quad \forall i < 0\}.$$
$$Spctr^{\geq 0} = \{X \in Spctr \mid \pi_i(X) = 0 \quad \forall i > 0\}.$$
(connective).

Def'n: The ∞-category of $\underline{\mathbb{E}_\infty\text{-rings}}$ is defined as.

$$Alg_{\mathbb{E}_\infty} := CAlg(Spctr^{\leq 0}).$$

As before, one has a functor

$$z: \underset{\text{is}}{CAlg(Ab)}. \longrightarrow CAlg(Spctr^{\leq 0}). \quad \text{s.t.}$$

CRings
- z is fully faithful.
- essential image of z is $\tau_{\leq 0}(CAlg(Spctr^{\leq 0}))$, i.e. the discrete objects in $\mathbb{E}_\infty$-rings.

In particular, for k a field we let

$$Alg_{\mathbb{E}_\infty, k} := CAlg(Spctr^{\leq 0})^{z(k)/-}$$

Thm: There exists a functor:
$$\Psi: CAlg_k \longrightarrow Alg_{\mathbb{E}_\infty, k} \quad \text{which is}$$
an equivalence. for any comm. ring k.

The functor comes from $u^R|_{Vect^{\leq 0}}: Vect^{\leq 0} \to Spctr^{\leq 0}$
where $u: Spctr \to Vect$ is the unit map in Cat_∞^{st} & (u, u^R).

we have

Before giving an idea of why this result we will need to discuss some modules over derived rings.

For completeness, before we do that we will consider a third model of derived rings.

Consider $Poly_k$ the ordinary cat of finitely generated polynomial algebras over k (a comm. ring), e.g. $k[x_1, \dots, x_n]$ are its objects.

We let:

$$\mathcal{P}_\Sigma(Poly_k) := \{ F : Poly^{op} \to Spc \mid F \text{ preserves finite products} \}.$$

Rk: For any ∞-cat. $\mathcal{C}$ w/ finite coproducts one can define $\mathcal{P}_\Sigma(\mathcal{C})$ as above. Moreover, $\mathcal{P}_\Sigma(\mathcal{C})$ is a model for the sifted completion of $\mathcal{C}$.

↳ formally adjoins all sifted colimits.

Def'n: A simplicial set (or ∞-cat) K is sifted if the diagonal functor $\delta : K \to K \times K$ is cofinal, i.e.

$\forall\ F : K \times K \to Spc$

$$\operatorname*{Colim}_{K \times K} F \xrightarrow{\simeq} \operatorname*{Colim}_{K \circ \delta} F.$$

E.g.: - any filtered category, (or ∞-category) is sifted.

- Δ^{op} is sifted.

Here are some properties of the construction $\mathcal{C} \mapsto \mathcal{P}_\Sigma(\mathcal{C})$.

Prop: (i) $\mathcal{P}_\Sigma(\mathcal{C})$ is cocomplete;

(ii) the natural Yoneda inclusion $z : \mathcal{C} \hookrightarrow \mathcal{P}_\Sigma(\mathcal{C})$ preserves coproducts;

(iii) the essential image of z are the compact & projective objects.

(iv) $\forall\ \mathcal{D}$ w/ sifted colimits, $\underbrace{\mathrm{Fun}_{\Sigma}}_{\text{(preserves sifted)}}(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Def'n: The category of simplicial commutative k-algebras. is defined as:

$$SCR_k := \mathcal{P}_\Sigma(Poly_k).$$

Rk: (i) let $\underline{SCR} := \underline{Fun}_\Sigma(Poly_k^{op}, Sets_\Delta)$ ordinary category of finite product-preserving functors. (Exercise: check this is equivalent to a naïve definition of $\underline{SCR}^{naive} := \underline{Fun}(\Delta^{op}, CRing)$.)

$\underline{SCR}$ has a simplicial model structure, i.e.
w.e. := $\{f: X \to Y \mid f(R): X(R) \to Y(R)$ is w.e. in $Sets_\Delta$ $\forall R \in Poly_k\}$.
fibrations := $\{$ ———— ———— — fibration ———— $\}$.

Then [HTT, Cor. 5.5.9.3] $\Rightarrow$ $SCR_k = \mathcal{P}_\Sigma(Poly_k)$
$\simeq$ $N^{hc}(\underline{SCR}_{cf})$. cofibrant-fibrant subcategory.

(ii) Certain references, e.g. Cesnavicius-Scholze call objects of SCR_k animated rings.
Let $\Theta_0 : Poly_k \to Alg_{\mathbb{E}_\infty, k}$ denote the inclusion of polynomial k-algebras into discrete objects of $Alg_{\mathbb{E}_\infty, k}$.
By construction of SCR_k one obtains a functor:
$$\Theta : SCR_k \to Alg_{\mathbb{E}_\infty, k}$$ which commutes w/ sifted colimits.

Thm: (i) Θ preserves limits & colimits.
(ii) Θ is conservative.
(iii) Θ has left & right adjoints.
(iv) If $\mathbb{Q} \subseteq k$, Θ is an equivalence. of ∞-categories.

We don't have quite enough theory to prove or give an idea why this result works just yet. Go back to discussion of monoidal ∞-cats. (Talk 9).